FROM
ALGEBRAIC COBORDISM
TO
MOTIVIC COHOMOLOGY
PART I

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We have seen that MGL is the universal oriented ring spectrum. Motivated by the analogous topological situation, it is therefore natural to ask:

Question. Does algebraic cobordism MGL carry the universal formal group law? More precisely, is the classifying morphism

$$L \to MGL_{(2,1)*}$$

of its formal group law an isomorphism[1]?

The aim of this sequence of talks is to give an affirmative answer to this question in the case where our base-scheme is a field of characteristic 0, and a partial answer for all fields.

1. Preliminaries

1.1. **Notation.** Let S be a Noetherian scheme of finite Krull dimension - in most parts of this series of talks, S will in fact be a field.

Write $Spc_*(S)$ for the category of *motivic spaces* (i.e. presheaves of pointed simplicial sets). Inside Spc_*, we have the bigraded spheres

$$S^{p.q} := S^{p-q} \wedge \mathbb{G}_m$$

From this, we can construct the symmetric monoidal category of symmetric $S^{2,1}$-spectra.
The categories $Spc_*(S)$ and Spt can be endowed with several different model structures obtained by left Bousfield localising model structures which are defined object- and levelwise. The subtle differences between these will not be crucial in this talk- the only fact we will use is that there exist model structures such that objectwise / levelwise monomorphisms are cofibrations.
Write

- $Map(X, Y)$ for the derived mapping space between two motivic spaces or spectra
- $[X, Y] = \pi_0(Map(X, Y))$ for its connected components
- $\pi_{p,q}(X) = [S^{p,q}, X]$ for the homotopy classes of maps from the motivic spheres into our space/spectrum X
- $\underline{\pi}_{p,q}(X)$ for the homotopy sheaves of X

[1]Here L denotes the usual Lazard ring.

1.2. **A reminder on MGL.** We will now briefly remind ourselves of the definition and basic properties of algebraic cobordism.
Write

- $\mathrm{Gr}(r,n)$ for the Grassmannian of r-planes in $\mathbb{A}^n_S$
- $E(r,n) \to \mathrm{Gr}(r,n)$ for its tautological bundle
- $\mathbb{P}^\infty := \mathrm{colim}(\mathbb{P}^n)$ for infinite projective space
- $\mathrm{Th}(E(1,\infty)) := \mathrm{colim}\,\mathrm{Th}(E(1,n)) \to \mathbb{P}^\infty$.

In order to define MGL as a highly structured ring spectrum, we now note that the action of Σ_n on $(\mathbb{A}^m)^n$ yields an action of Σ_n on the whole diagram

$$\begin{array}{ccccc} E(n,n) & \longrightarrow & E(n,2n) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \\ \mathrm{Gr}(n,n) & \longrightarrow & \mathrm{Gr}(n,2n) & \longrightarrow & \dots \end{array}$$

and thus on the motivic space

$$MGL_n = \operatorname*{colim}_{m \geq n} \mathrm{Th}(E(n,mn))$$

There are canonical $\Sigma_n \times \Sigma_p$-equivariant maps

$$MGL_n \wedge MGL_p \to MGL_{n+p}$$

which give the structure maps of a symmetric spectrum

$$MGL := \operatorname*{colim}_{n} \Sigma^{-2n,-n}\Sigma^\infty MGL_n$$

and even of an algebra object in $Spt(S)$.

The canonical map $\Sigma^{-2,-1}\Sigma^\infty\,\mathrm{Th}(E(1,\infty)) \to MGL$ gives a Thom class

$$th^{MGL} \in MGL^{2,1}(Th(E(1,\infty)))$$

of the universal line bundle and therefore orients MGL. We conclude that MGL is a *highly structured oriented ring spectrum*.

2. The Motivic Quillen Theorem

The orientation of MGL gives rise to a formal group law and thus to a homomorphism $L \to MGL_{(2,1)*} = \bigoplus_i MGL_{2i,i}$. The main aim of this sequence of talks is to prove the following theorem:

Theorem 1. (Hoyois,Hopkins-Morel) Let k be a field of characteristic exponent [2] c and consider algebraic cobordism $MGL \in Spt(k)$.
Then the canonical map

$$\theta : L\left[\frac{1}{c}\right] \to MGL_{(2,1)*}\left[\frac{1}{c}\right]$$

is an isomorphism.

For simplicity, we shall assume that $\mathrm{char}\, k = 0$ for the rest of this talk.

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[2]The characteristic exponent of a field is defined by $c = \begin{cases} 1 & \text{if } \mathrm{char}\, k = 0, \\ \mathrm{char}\, k & \text{otherwise} \end{cases}$.

2.1. **The Strategy of the Proof.** We will now give an outline of the proof and explain several initial reductions.

2.1.1. *Injectivity of* θ. The injectivity of the comparison map θ will, in analogy with the classical setting, follow by first computing $H\mathbb{Z}_{*,*}(MGL)$ and then verifying that the composite

$$L \to MGL_{(2,1)*} \xrightarrow{\text{Hurewicz}} H\mathbb{Z}_{*,*}(MGL)$$

of the classifying map with the Hurewicz is injective. The relevant details will be provided in the subsequent talks.

2.1.2. *Surjectivity of* θ. This is the difficult part of the proof. We will start off by reducing the problem of proving surjectivity of θ to a claim about the comparison map between algebraic cobordism and motivic cohomology.

The Initial Reduction. In order to prove that θ is surjective, we will proceed roughly as follows:

- Restrict attention to a certain degree $L_n \to MGL_{2n,n}$, assuming that the result holds true in all smaller degrees
- Choose "adequate" generators[3] a_1, $a_2, \ldots \in L \subseteq MGL_{(2,1)*}$ with
$$|a_i| = (2i, i)$$
- Prove that $(L/a_1, \ldots, a_n)_n \to (MGL/(a_1, \ldots, a_n))_{2n,n}$ is an isomorphism
- Work backwards by proving that if $(L/a_1, \ldots, a_{k+1})_n \to (MGL/(a_1, \ldots, a_{k+1}))_{2n,n}$ is surjective, then so is $(L/a_1, \ldots, a_k)_n \to (MGL/(a_1, \ldots, a_k))_{2n,n}$.

We will start by addressing the very last point. Fix "adequate" generators

$$a_i \in L_{2i,i} \subset MGL_{2i,i}$$

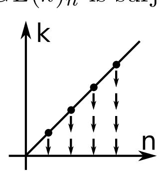
and set [4]

$$L(k) = L/(a_1, \ldots, a_k), \qquad MGL(k) = MGL/a_1, \ldots, a_k.$$

Lemma 2. Assume that $L(n)_n \to MGL(n)_n$ is surjective for all n.

Then for all $k \leq n$:

$$L(k)_n \to MGL(k)_n \text{ is surjective. } (\star_{k,n})$$



Proof. We proceed by induction. Assume that $(\star)_{\ell,m}$ is true whenever $m < n$ or $m = n$ and $\ell > k$. Consider the diagram

[3] "Adequacy" is a certain technical condition which we will define later.

[4] Here we abuse notation - we really picked representatives $\overline{a}_i : S^{2i,i} \to MGL$ of the various classes a_i.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L(k)_{n-k-1} & \xhookrightarrow{a_{k+1}} & L(k)_n & \twoheadrightarrow & L(k+1)_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} & & \\
 & & MGL(k)_{n-k-1} & \xrightarrow{a_{k+1}} & MGL(k)_n & \longrightarrow & MGL(k+1)_n & &
\end{array}$$

By induction, the maps a and b are surjective. So c is surjective. □

Hence in order to prove the motivic Quillen theorem, it is enough to show injectivity of θ and that $L(n)_n \to MGL(n)_n$ is surjective for all n. In order to reduce this statement further, we will need the homotopy $t-$structure, which we will briefly review now.

Digression: The Homotopy $t-$structure

In this section, we go back to the case where our base scheme S is any Noetherian finite-dimensional scheme. This extra generality is not strictly necessary for our purposes, but will allow us to phrase some results about MGL over a general base scheme.

Definition 3. (Hoyois) The category $\mathcal{SH}(S)_{\geq d}$ of *d-connective spectra* is defined to be the subcategory of $\mathcal{SH}(S)$ generated under homotopy colimits and extensions by all spectra of the form

$$\Sigma^{p,q}\Sigma^\infty_+ X$$

for $X \in Sm_{/S}$ and $p - q \geq d$.

One can prove:

Lemma 4. $\mathcal{SH}(S)_{\geq 0}$ is the nonnegative part of a unique t-structure.

If S is a field, then we have an a priori different $t-$structure due to Morel, who defined a motivic spectrum E to be d-connective if $\underline{\pi}_{p,q}(E) = 0$ for all $p - q < d$. Fortunately, we have:

Theorem 5. (Hoyois) In the case where $S = k$ is a field, the two t-structures agree.

We recall the following result, which has essentially already been covered in the talk about $t-$structures:

Theorem 6. For k a field, $X \in \mathrm{Sm}_{/k}$ a smooth scheme, p, $q \in \mathbb{Z}$ integers, $E \in \mathcal{SH}(k)$, and $d > (p - q) + \dim X$, we have:

$$[\Sigma^{p,q}\Sigma^\infty_+ X, E_{\geq d}] = 0.$$

Back to the Motivic Quillen Theorem

Assume again that $S = k$ is a field of characteristic 0.

We now return to our problem of proving surjectivity of θ, which we had reduced to the claim that

$$L(n)_n \to MGL(n)_{2n,n}$$

is surjective for all n.

Since the generator a_m lives in bidegree $(2m, m)$, we intuitively expect that dividing out a_m on both sides will not alter the statement.

In order to prove that this is indeed the case, we need the following result which we will prove very soon:

Theorem 7. MGL is connective.

Corollary. *For $k \geq n$, the map $MGL(k)_n \to MGL(k+1)_n$ is an isomorphism.*

Proof. Since the spectrum $\Sigma^{2(k+1),(k+1)} MGL(k)$ is $(k+1)$-connective, theorem 6 shows that $\pi_{2n,n}(\Sigma^{2(k+1),(k+1)} MGL(k)) = 0$ and that $\pi_{2n-1,n}(\Sigma^{2(k+1),(k+1)} MGL(k)) = 0$. We apply this to the long exact sequence associated to the cofibre sequence

$$\Sigma^{2(k+1),k+1} MGL(k) \xrightarrow{a_{k+1}} MGL(k) \to MGL(k+1)$$

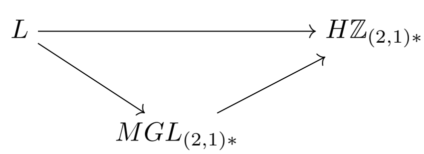
and obtain the result. □

Hence in order to prove the motivic Quillen theorem, it is enough to show that

$$\mathbb{Z} = L/(a_1, a_2, \ldots) \to (MGL/(a_1, a_2, \ldots))_{(2,1)*}$$

is an isomorphism.

Note that the representing spectrum $H\mathbb{Z}$ of motivic cohomology is oriented, which gives a map of ring spectra $MGL \to H\mathbb{Z}$ *from algebraic cobordism to motivic cohomology.* The composite



then classifies the additive formal group law, which shows that all generators of the Lazard ring L must go to zero. This observation gives rise to a map

$$MGL/(a_1, a_2, \ldots) \to H\mathbb{Z}$$

As

$$(H\mathbb{Z})_{2n,n} = \begin{cases} 0 & \text{if } n \neq 0, \\ \mathbb{Z} & \text{otherwise,} \end{cases}$$

we see that surjectivity in the motivic Quillen theorem is implied by:

Theorem 8 (Hoyois, Hopkins–Morel)**.** The map $MGL/(a_1, a_2, \ldots) \xrightarrow{f} H\mathbb{Z}$ is an equivalence. [5]

[5] This statement holds over any field of characteristic zero, and the same proof goes through if we invert the characteristic exponent of our base field.

The connectivity of MGL. For the rest of this talk, let S be again a general Noetherian finite-dimensional base-scheme.

The aim of this section is to prove one of the first main ingredients to the proof of theorem 8, namely:

Theorem 9. The algebraic cobordism spectrum MGL is connective.

Proof. The strategy is to first produce a map from an obviously connective spectrum T into MGL, and then show that this map induces an equivalence after applying the functor $(-)_{\leq 0}$.

There is an obvious candidate for T, namely a desuspension of the first piece $Th(E(1,2))$ of the first component MGL_1 of the spectrum MGL:

$$T := \Sigma^{-2,-1}\Sigma^{\infty}\,\mathrm{Th}(E(1,2)) \to MGL$$

Remark 10. It is immediate that $\mathrm{Th}(E(1,2))$ is equivalent to the cofibre of the Hopf map $h : \mathbb{A}^2 - \{0\} \to \mathbb{P}^1$:

$$\begin{array}{ccc} E(1,2) - \mathbb{P}^1 & \hookrightarrow & E(1,2) \\ \downarrow \sim & & \downarrow \sim \\ \mathbb{A}^2 - 0 & \xrightarrow{h} & \mathbb{P}^1 \end{array}$$

Since T is connective, it suffices to prove:

Theorem 11. The map

$$T_{\leq 0} \to MGL_{\leq 0}$$

is an equivalence.

Proof. Since the functor $(-)_{\leq 0}$ preserves filtered homotopy colimits, it is enough to show that each of the maps

$$\Sigma^{\infty}T \xrightarrow{f_r} \Sigma^{-2r,-r}\Sigma^{\infty}MGL_r$$

induces an equivalence after applying $(-)_{\leq 0}$, i.e. that the map

$$\Sigma^{2(r-1),(r-1)}\,\mathrm{Th}(E(1,2)) \to MGL_r$$

induces an equivalence after applying $(\Sigma^{\infty}(-))_{\leq r}$ (such a map of motivic spaces is called *stably* $r-$*connective*). But this map is defined to be the composite

$$\begin{array}{l} \Sigma^{2(r-1),(r-1)}\,\mathrm{Th}(E(1,2)) \\ \qquad\downarrow \\ \Sigma^{2(r-1),(r-1)}\,\mathrm{Th}(E(1,3)) \\ \qquad\downarrow \\ \qquad\cdots \\ \qquad\downarrow \\ \Sigma^{2(r-1),(r-1)}MGL_1 \longrightarrow \Sigma^{2(r-2),(r-2)}MGL_2 \longrightarrow \cdots \longrightarrow MGL_r \end{array}$$

and it is sufficient to show that each of the individual maps is stably $r-$connective. Unfortunately, the spaces occuring in the horizontal part of this diagram are inconvenient to work with using algebraic-geometric methods since they classify subspaces of infinite affine space.
This problem can be circumvented by considering commutative squares of the form

$$\begin{array}{ccc} \Sigma^{2,1}\,\mathrm{Th}(E(k,\ell)) & \longrightarrow & \mathrm{Th}(E(k+1,\ell+1)) \\ \downarrow & & \downarrow \\ \Sigma^{2,1}\,\mathrm{Th}(E(k,\ell+1)) & \longrightarrow & \mathrm{Th}(E(k+1,\ell+2)) \end{array}$$

The vertical maps have been defined before.
The top horizontal map lives above the map of Grassmannians which takes a $k-$space in $\mathbb{A}^{\ell}$ sends it to the $(k+1)$-space in $\mathbb{A}^{\ell+1}$ obtained by "adding" the "last" basis vector to it. The bottom horizontal map has an analogous description. We can therefore add all these squares to the above diagram and obtain:

$$\begin{array}{ccccccc} \Sigma^{2(r-1),(r-1)}\,\mathrm{Th}(E(1,2)) & \longrightarrow & \Sigma^{2(r-2),(r-2)}\,\mathrm{Th}(E(2,3)) & \longrightarrow & \dots & \longrightarrow & \mathrm{Th}(E(r,r+1)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^{2(r-1),(r-1)}\,\mathrm{Th}(E(1,3)) & \longrightarrow & \Sigma^{2(r-2),(r-2)}\,\mathrm{Th}(E(2,4)) & \longrightarrow & \dots & \longrightarrow & \mathrm{Th}(E(r,r+2)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & & \dots & & \dots & & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^{2(r-1),(r-1)}MGL_1 & \longrightarrow & \Sigma^{2(r-2),(r-2)}MGL_2 & \longrightarrow & \dots & \longrightarrow & MGL_r \end{array}$$

We see that $\Sigma^{2(r-1),(r-1)}\,\mathrm{Th}(E(1,2)) \to MGL_r$ is also the composite of the top horizontal maps and the right vertical maps. The top horizontal maps are all of the form $\Sigma^{2(r-(k-1)),r-(k-1)}\left(\Sigma^{2,1}\,\mathrm{Th}(E(k-2,k-1)) \to \mathrm{Th}(E(k-1,k))\right)$ for $k = 3,4,...,r+1$. We therefore see that in order to prove theorem 11, it is enough to prove the following result:

Theorem 12. We have:

- Horizontal: $\Sigma^{2,1}\,\mathrm{Th}(E(s-1,t-1)) \to \mathrm{Th}(E(s,t))$ is stably $2s$-connective,
- Vertical: $\mathrm{Th}(E(r,t-1)) \to \mathrm{Th}(E(r,t))$ is stably t-connective.

Proof. We will only prove the vertical assertion, the horizontal one is proven similarly. We will first need to understand the connectivity of the underlying map

$$\mathrm{Gr}(r,n-1) \to \mathrm{Gr}(r,n)$$

of Grassmannians. This seems hard - let us first contemplate on spaces whose connectivity we do know:

- The bigraded sphere $S^{p,q} = (S^1)^{p-q} \wedge \mathbb{G}_m^q$ is stably $(p-q)-$connective.
- The Thom space of $(\mathbb{A} \to pt)$ is stably $n-$*connective* as

$$\mathrm{Th}(\mathbb{A}^n \to pt) = \mathrm{Th}(\mathbb{A}^1 \to pt)^{\wedge n} \cong S^{2n,n}$$

- The Thom space of the trivial bundle $(\mathbb{A} \times U \to U)$ is stably $n-$connective as
$$\mathrm{Th}(\mathbb{A}^n \times U \to U) = \Sigma_+^{2n,n}(U)$$
- The Thom space of any rank n bundle $E \to X$ is stably $n-$connective. Indeed, we can pick a cover $\{U_\alpha\}$ of X over which E trivialises and then show that
$$\mathrm{Th}(E) = hocolim \left(... \substack{\rightarrow\\\rightarrow\\\rightarrow} \bigvee \mathrm{Th}(E|_{U_\alpha \cap U_\beta}) \rightrightarrows \bigvee \mathrm{Th}(E|_{U_\alpha}) \right)$$
Since all terms in this diagram are stably $n-$connective, so is $\mathrm{Th}(E)$.
- By the purity theorem, this implies that whenever $Z \to X$ is a closed embedding of smooth schemes over S, the quotient
$$X/(X - Z) \cong \mathrm{Th}(N_{X,Z})$$
is stably $codim(Z, X)$-connective.

Back to our original aim of computing the connectivity of $\mathrm{Gr}(r, n)/\mathrm{Gr}(r, n-1)$. We naturally want to write $\mathrm{Gr}(r, n-1)$ as the complement of a closed subscheme and then use the previous argument - unsurprisingly, this will not work since the complement of $\mathrm{Gr}(r, n-1)$ is not closed. However, we can get around this problem by "thickening up" $\mathrm{Gr}(r, n-1)$ by a weak equivalence.

Indeed, recall the map $j : \mathrm{Gr}(r-1, n-1) \to \mathrm{Gr}(r, n)$ which "adds on the last coordinate" from before and note that the natural "projection" map
$$\mathrm{Gr}(r, n) - im(j) \to \mathrm{Gr}(r, n-1)$$
is a vector bundle of rank r. We conclude:

Lemma 13. The motivic space
$$\mathrm{Gr}(r, n)/\mathrm{Gr}(r, n-1)$$
is stably $(n-r)-$connective.

Proof. We use the weak equivalence
$$\mathrm{Gr}(r, n)/\mathrm{Gr}(r, n-1) \cong \mathrm{Gr}(r, n)/(\mathrm{Gr}(r, n) - \mathrm{im}(j))$$
to compute that the connectivity is given by
$$\mathrm{codim}\,(\mathrm{Gr}(r, n), \mathrm{Gr}(r-1, n-1)) = r(n-r) - (r-1)((n-1) - (r-1)) = n - r$$
□

We now want to compute the connectivity of $\mathrm{Th}(E(r,n))/\mathrm{Th}(E(r,n-1))$ by rewriting it as a quotient of a scheme by the complement of a closed subscheme. The first idea is to express this double quotient as a single quotient:

$$\begin{array}{ccccc}
E(r,n-1)-\mathrm{Gr}(r,n-1) & \longrightarrow & E(r,n)-\mathrm{Gr}(r,n) & & \\
\downarrow & & \downarrow & & \\
E(r,n-1) & \longrightarrow & E(r,n) & \longrightarrow & E(r,n)/\left(E(r,n-1)\coprod_{\cdots} E(r,n)-\mathrm{Gr}(r,n)\right) \\
\downarrow & & \downarrow & & \downarrow \wr \\
\mathrm{Th}(E(r,n-1)) & \longrightarrow & \mathrm{Th}(E(r,n)) & \longrightarrow & \mathrm{Th}(E(r,n))/\mathrm{Th}(E(r,n-1))
\end{array}$$

But we encounter the same problem as before: It is geometrically clear that the complement of $\left(E(r,n-1)\coprod_{E(r,n-1)-\mathrm{Gr}(r,n-1)}(E(r,n)-\mathrm{Gr}(r,n))\right)$ in $E(r,n)$ is *not* closed - it is exactly $\mathrm{Gr}(r,n)-\mathrm{Gr}(r,n-1)$.

We resolve this issue using the same trick as before, namely by enlarging the schemes we quotient out by.

First note:

- $E(r,n-1)$ is the restriction of $E(r,n)\big|_{\mathrm{Gr}(r,n)\text{-im(j)}}$ along the zero section

$$z : \mathrm{Gr}(r,n-1) \to \mathrm{Gr}(r,n)\text{-im(j)}$$

- $E(r,n)\big|_{\mathrm{Gr}(r,n)\text{-im(j)}}$ is the pullback of $E(r,n-1)$ along the projection

$$p : \mathrm{Gr}(r,n)\text{-im(j)} \to \mathrm{Gr}(r,n-1)$$

We can now show:

Lemma 14. The horizontal maps in the diagram

$$\begin{array}{ccc}
E(r,n-1)-\mathrm{Gr}(r,n-1) & \overset{p_2}{\underset{\sim}{\rightleftarrows}} & E(r,n)\big|_{\mathrm{Gr}(r,n)\text{-im(j)}}-(\mathrm{Gr}(r,n)\text{-im(j)}) \\
\downarrow & & \downarrow \\
E(r,n-1) & \overset{p_1}{\underset{\sim}{\rightleftarrows}} & E(r,n)\big|_{\mathrm{Gr}(r,n)\text{-im(j)}} \\
\downarrow & & \downarrow \\
\mathrm{Gr}(r,n-1) & \overset{p}{\underset{z}{\rightleftarrows}} & \mathrm{Gr}(r,n)\text{-im(j)}
\end{array}$$

are weak equivalences.

Proof. Indeed, the map p is a vector bundle. Since p_1 is the pullback of this vector bundle p, it is itself a vector bundle, and the same applies to p_2. $\square$

We can now use this result to modify the previous presentation of the quotient of Thom spaces $\mathrm{Th}(E(r,n))/\,\mathrm{Th}(E(r,n-1))$ to

$$E(r,n)/\left(E(r,n)\big|_{\mathrm{Gr}(r,n)\text{-im(j)}} \coprod_{E(r,n)\big|_{\mathrm{Gr}(r,n)\text{-im(j)}}-(\mathrm{Gr}(r,n)\text{-im(j)})} (E(r,n)-\mathrm{Gr}(r,n)) \right)$$

which is just

$$E(r,n)/(E(r,n)-im(j))$$

and is therefore stably $(codim(im(j),E(r,n))=n)$-connective.

This finishes the proof of (the first part of) theorem 12, □

hence we have established theorem 11, □

and since T is obviously connective, this concludes the proof of the connectivity of algebraic cobordism asserted in 9. □